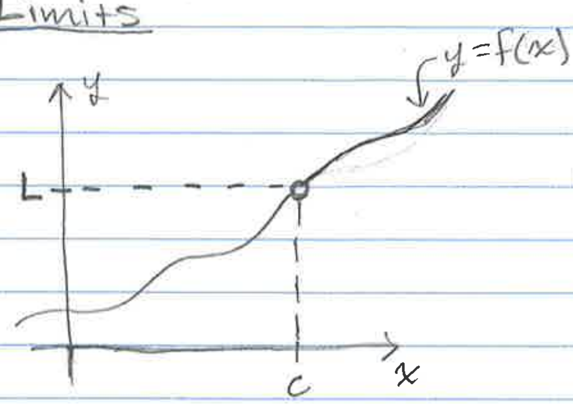
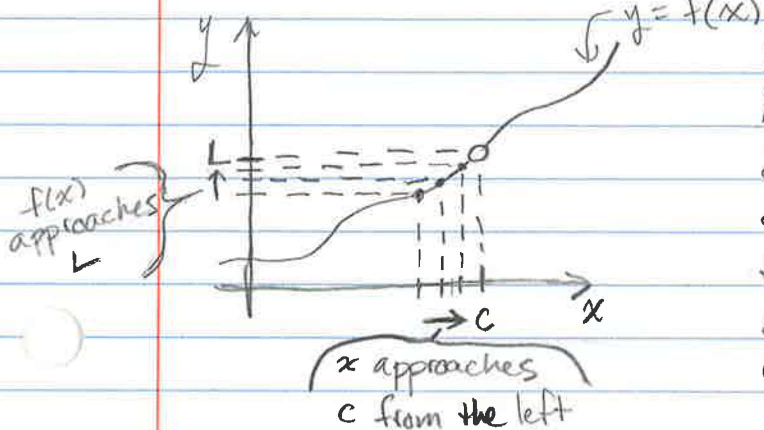
Single Variable Calc.

start video 1

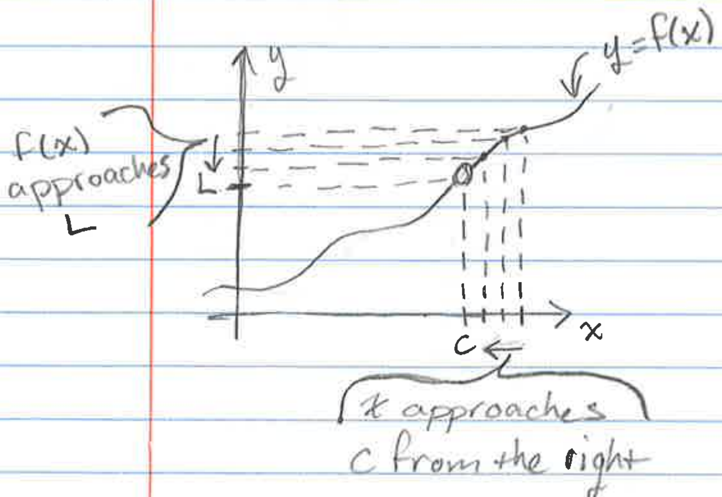
Limits



What does f(x) approach when x approaches c?



As x approaches c from the left (i.e., if we look at values of x less than c that get closer and closer to c) the value of f(x) appears to approach some value (here labeled L).



As x approaches c from the right (i.e., if we look at values of x greater than c that get closer and closer to c) the value of f(x) appears to approach the same value as before, here labeled L.

We call the value that f(x) is approaching as x approaches c the <u>limit</u> of f(x) as x approaches c.

So here we would say, "the limit of f(x) as x approaches c is L", and we write $\boxed{\lim_{x \to c} f(x) = L}$

end video 1

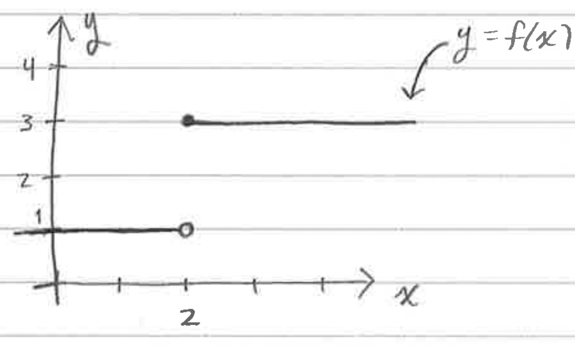
Note: in the video, the function given, $f(x)$, is not defined at $x = c$ (this is what the open circle means). So, $f(c) \neq L$ ($f(c)$ does not equal anything as it is undefined), but we do have $\lim_{x \to c} f(x) = L$.

you might think of this as meaning $f(x) \to L$ as $x \to c$ (i.e., $f(x)$ approaches L as x approaches c).

In order for the limit to exist, the function must approach the same value as x approaches c from both sides.

Example

$$f(x) = \begin{cases} 1, & x < 2 \\ 3, & x \geq 2 \end{cases}$$



What is the limit of $f(x)$ as x approaches 2?

$\lim_{x \to 2} f(x) = ?$

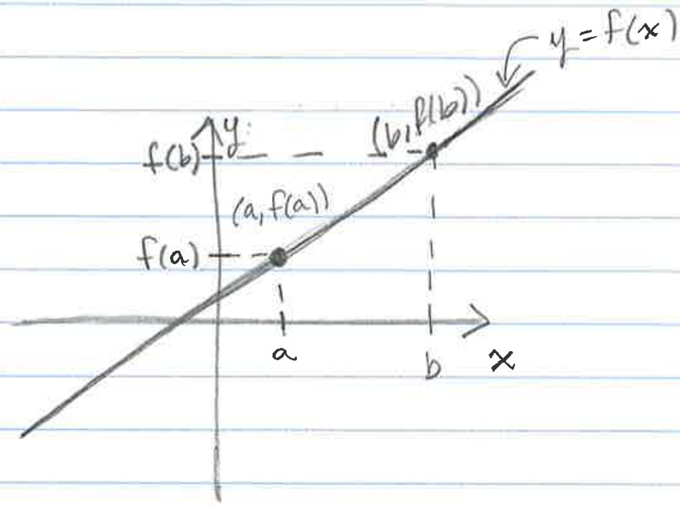
As x approaches 2 from the left, $f(x)$ approaches the value 1.

As x approaches 2 from the right, $f(x)$ approaches the value 3.

Since $f(x)$ approaches different values when x approaches 2 from the left and right, the limit here does not exist: $\lim_{x \to 2} f(x)$ DNE

Start video 2

Derivatives



How do we find the slope of a line?

Take two points on the line and compute

$$\frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

$$\text{change in } y = \Delta y = f(b) - f(a)$$
$$\text{change in } x = \Delta x = b - a$$

$$\Rightarrow \boxed{\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}}$$

Can we generalize this to a curve?

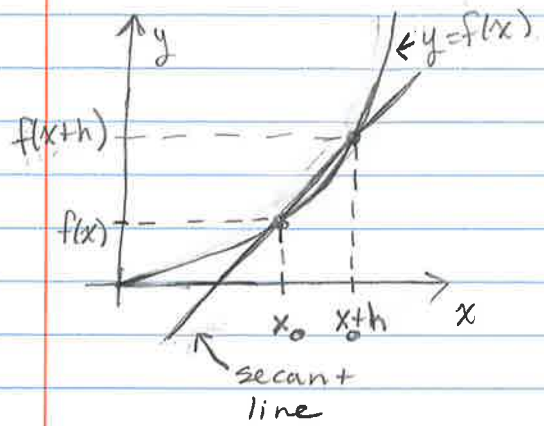
On a line, the slope is the same no matter where you are on the line, but the "slope" of a curve can vary depending upon where you are on the curve.

Example:

same curve, different "slopes"

Recall: a tangent line to a curve at a certain point is the straight line that "just touches" the curve at that point. Here, we are using the slopes of the tangent lines to determine the slope of the curve at different points

How then can we compute the slope of a curve at a particular point on that curve?



Suppose we want the slope of this curve at the indicated x_0.

We know from finding the slope of a line that we need 2 points, so let's take the point that is h units to the right of x_0, or x_0+h.

If we use these points to calculate the slope, we will actually be calculating the slope of the secant line that passes through the points $(x, f(x))$ and $(x+h, f(x+h))$ on the curve.

Recall: a secant line is simply a line passing through at least two points on a curve.

So, the slope of the secant line drawn above is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{x_0+h - x_0} = \boxed{\frac{f(x_0+h)-f(x_0)}{h}}$$

How can we use this to find the slope of the curve at x_0?
The closer x_0+h is to x_0, the closer the slope of the secant line will be to the "slope" of the curve (or the slope of the tangent line to the curve at x_0).
- We can make x_0+h approach x_0 by having h approach 0

So, the slope of the curve at x_0 (or the slope of the tangent line to the curve at x_0) is $\boxed{\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}} = f'(x_0)$

We call this the <u>derivative of $f(x)$ at x_0</u>, and denote it $f'(x_0)$.

end video 2

Since the slope of the tangent line to a curve depends on where you are on the curve, the derivative of $f(x)$ is itself a function of x

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

<u>Important take away</u>: the derivative $f'(x)$ of a function $f(x)$ at any point x is equal to the slope of the tangent line to $f(x)$ at x.

Differentiability

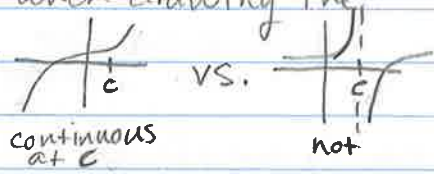
Does the function have a derivative at a certain point?
OR where does a function's derivative exist?

Note: in this video we'll be using a definition for the derivative that looks slightly different but is equivalent to the one we found in the last video.

Previous video definition: $f'(c) = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$

This videos definition: $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

For this video, we will also need the concept of continuity. Intuitively, a function is continuous at a point if you do not need to pick up your pencil when drawing the function through that point (e.g. vs.)



Definition: A function $f(x)$ is continuous at $x = c$ if and only if $\lim\limits_{x \to c} f(x) = f(c)$.

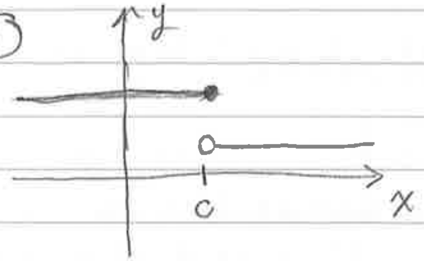
Now on to the important info from this video:

Start video 3 If f is differentiable at $x = c$, then f is continuous at $x = c$.

If f is not continuous at $x = c$, then f is not differentiable at $x = c$.

Example:

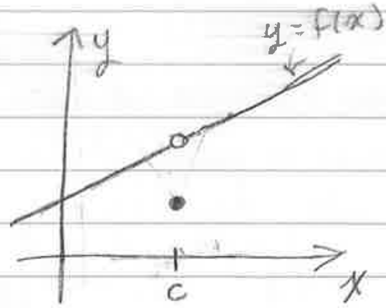
(1)



f is not continuous at c because $\lim_{x \to c} f(x)$ does not exist.

f is also not differentiable at c.
As $x \to c$ from the left $\frac{f(x)-f(c)}{x-c} = 0$.
But, as $x \to c$ from the right $\frac{f(x)-f(c)}{x-c}$ approaches $-\infty$.

(2)



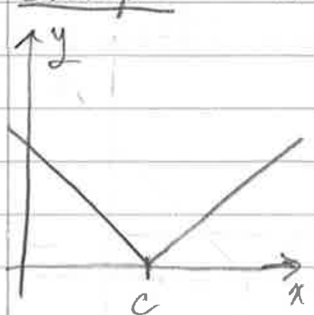
Note: the value of f(c) is given by the colored in circle

f is not continuous at c because $\lim_{x \to c} f(x) \neq f(c)$

f is also not differentiable at c.
As x approaches c from either side, $\frac{f(x)-f(c)}{x-c}$ approaches either + or $- \infty$, i.e., not a finite value.

If f is continuous at c, is it necessarily differentiable at c?

Example: $f(x) = |x-c|$, $\lim_{x \to c} f(x) = 0 = f(c)$ continuous at c!



as $x \to c$ from the left, $\frac{f(x)-f(c)}{x-c} = -1$

as $x \to c$ from the right $\frac{f(x)-f(c)}{x-c} = 1$

end video 3

So, $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ does not exist. Not differentiable at c!

Different Notation for Derivatives

Thus far we've been using the notation $f'(x)$ to denote the derivative of $f(x)$.

There is another commonly used notation that we will see a lot, which is $\frac{df}{dx}$.

This notation is actually more intuitive.
Recall, the derivative of a function f tells you the slope of f at any point x

$$\text{derivative of } f = \frac{\text{change in } f}{\text{change in } x} = \frac{df}{dx}$$

df and dx are called differentials and you can think of them as representing the size of the change in f (df) and the size of the change in x (dx).

$\frac{df}{dx}$ is a <u>function</u> (so, not actually a fraction)

but the notation is suggestive of how we should think about derivatives.

Sometimes people will also write $y = f(x)$ and then denote the derivative of f as $\frac{dy}{dx} = \frac{df}{dx} = f'(x)$

Derivative Rules

Let c be a constant

$f(x)$	$f'(x) = df/dx$
c	0
x^c	cx^{c-1}
c^x	$c^x \ln(c)$
$\log_c(x)$	$\frac{1}{x\ln(c)}$, $x > 0$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$, $x \neq \frac{n\pi}{2}$ for n = odd integer
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$, $x \neq \pm 1$
$\cos^{-1}(x)$	$\frac{-1}{\sqrt{1-x^2}}$, $x \neq \pm 1$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$

Derivative Rules

Let c be a constant and $f(x)$, $g(x)$ differentiable functions

Constant Multiple Rule	$\frac{d}{dx}(cf(x)) = cf'(x)$
Sum Rule	$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$
Difference Rule	$\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$
Product Rule	$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x)+f(x)g'(x)$
Quotient Rule	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$
Chain Rule	$\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$

Chain Rule

$$\boxed{h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))\,g'(x)}$$

start video 4

Consider $h(x) = (\sin(x))^2$

What is $h'(x) = \dfrac{dh}{dx} = ?$ Use Chain rule!

Thought experiment:

$\dfrac{d}{dx}[x^2] = 2x$, $\dfrac{d}{da}[a^2] = 2a$ (so not a constant here)

a is the variable you're taking the derivative with respect to!

$$\frac{d}{d(\sin(x))}\left[(\sin(x))^2\right] = 2\sin(x)$$

By chain rule

$$h'(x) = \frac{dh}{dx} = \underbrace{2\sin x}_{\substack{\text{derivative of}\\ \text{outer function}\\ \text{with respect to inner}}} \underbrace{\cos x}_{\substack{\text{times derivative}\\ \text{of inner function}}}$$

$$\frac{dh}{dx} = \frac{d\left[(\sin(x))^2\right]}{dx} = \frac{d\left[(\sin(x))^2\right]}{d(\sin(x))} \cdot \frac{d[\sin(x)]}{dx} = 2\sin(x)\cos(x)$$

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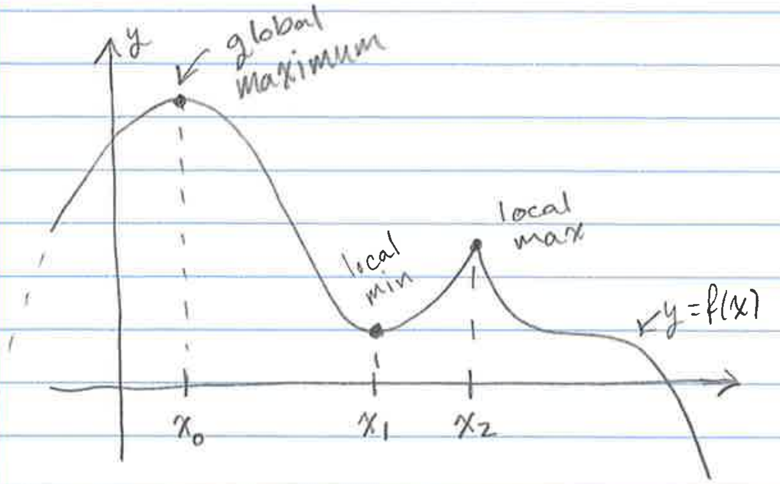
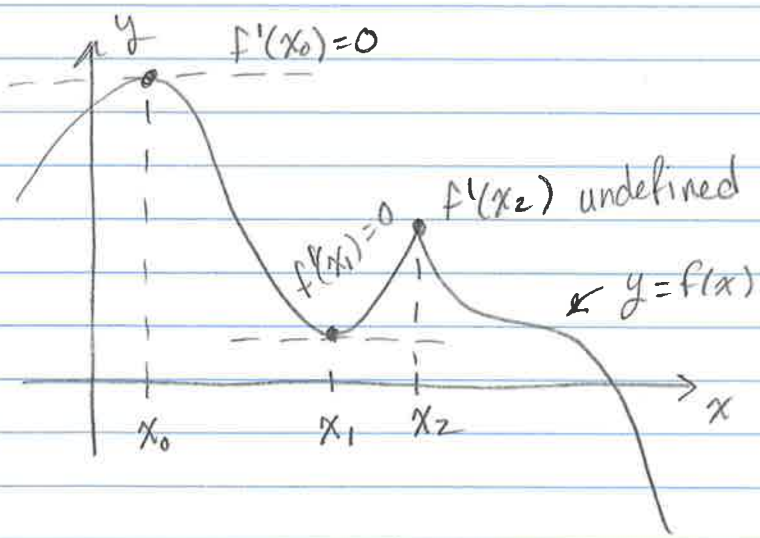
Local and Global Extrema and Critical Points



No global minimum

x_1 is a local min b/c $f(x_1)$ is smaller than $f(x)$ for any x in a region around x_1.

x_2 is a local max b/c $f(x_2)$ is larger than $f(x)$ for any x in a region around x_2.



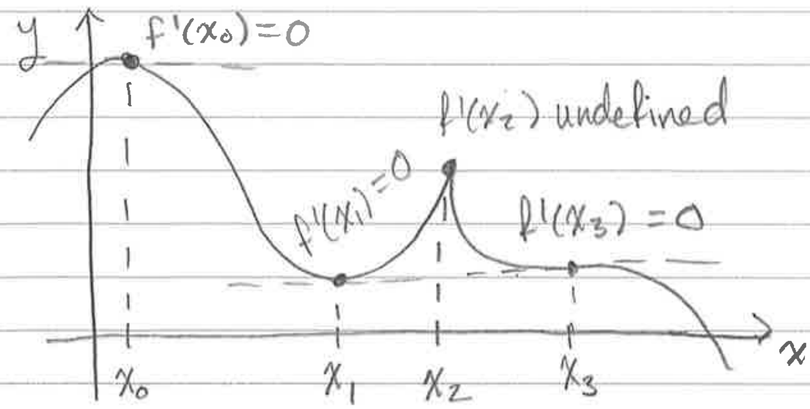
Recall: the derivative of a function f at a point $\hat{x}$ is equal to the slope of the tangent line to f at $\hat{x}$.

If a is not an end point and there is a local or global min or max at $x = a$ then either

$f'(a) = 0$ or

$f'(a)$ is undefined

Points a where $f'(a) = 0$ or $f'(a)$ is undefined are called <u>critical points</u>.



Critical points: $\underbrace{x_0, x_1, x_2}_{\text{min or max}}, x_3$ ← not a min or max

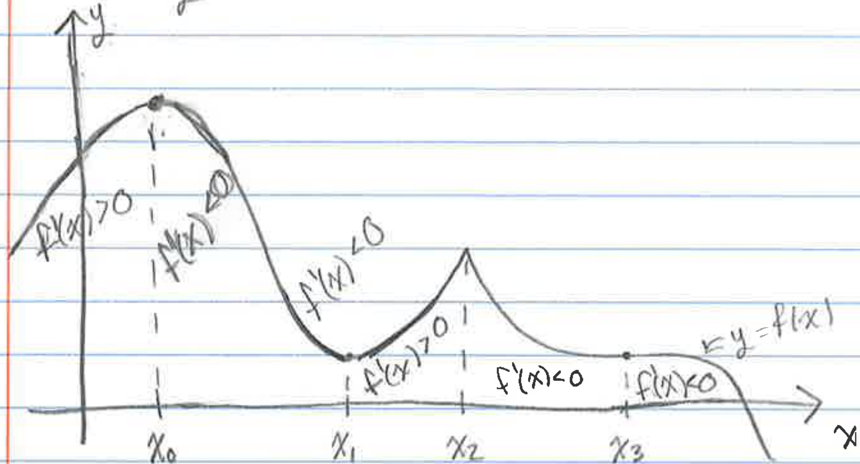
If a is a min or max (local or global) and is not an end point, then a must be a critical point.

end video 5

Just because a is a critical point does not necessarily mean it is a min or max!

start video 6

Finding Relative Extrema



the slope of f as we approach x_0 from the left is positive $\Rightarrow f'(x) > 0$ as we approach x_0 from the left

the slope of f as we approach x_0 from the right is negative $\Rightarrow f'(x) < 0$ as we approach x_0 from the right

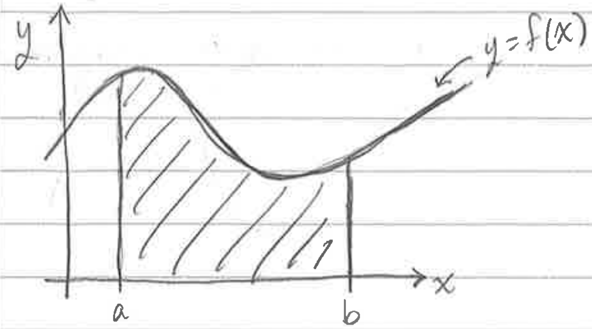
> If a is a critical point, then it is a max if $f'(x)$ switches signs from + to − as we cross $x = a$.

Test x_3 (not a max). $f'(x)$ is negative as we approach $x = x_3$ from the left. Once we cross x_3, $f'(x)$ is still negative $\Rightarrow x_3$ is not a max.

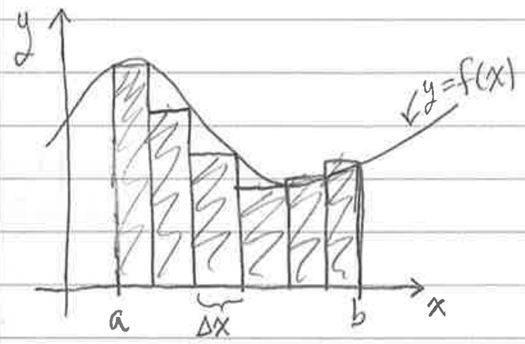
> If a is a critical point then it is a min if $f(x)$ switches signs from − to + as we cross $x = a$.

end video 6

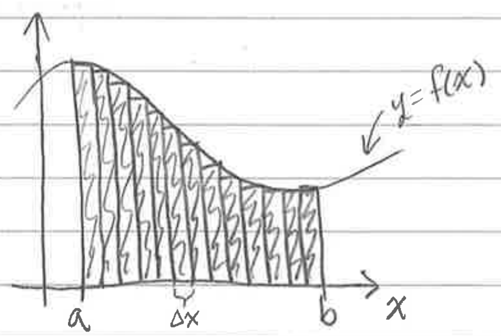
Integrals



What is the area under this curve (i.e. between this curve and the x-axis) between $x=a$ and $x=b$?



Can approximate it by cutting it up into a bunch of small rectangles w/ width Δx and adding up the areas of the rectangles: $\sum_{\text{rectangles}} (\text{height})\,\Delta x$



The appoximation gets better the smaller Δx is $\Rightarrow$ if we take the limit as $\Delta x \to 0$ we can get the exact value of the area under the curve between $x=a$ and $x=b$. We call this the <u>integral</u>.

$$\lim_{\Delta x \to 0} \sum_{\text{rectangles}} (\text{height})\,\Delta x = \int_a^b \text{height}(x)\,dx$$

What is the height of the curve at x?

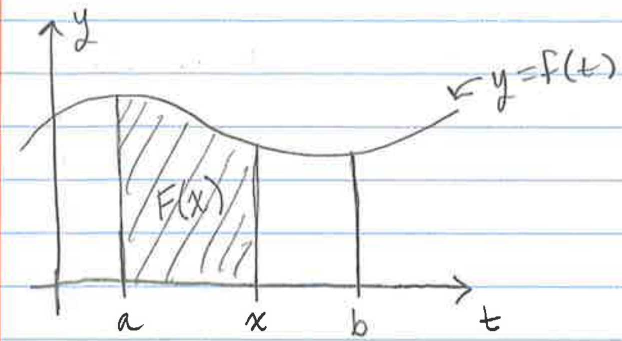
$$\text{height}(x) \stackrel{!}{=} f(x)$$

$$\begin{array}{c}\text{area under the} \\ \text{curve from } x=a \\ \text{to } x=b\end{array} = \int_a^b f(x)\,dx$$

"the integral from a to b of f with respect to x."

start video 7

Fundamental Theorem of Calculus 1



f is continuous on [a,b] (this ensures us that the integral will exist).

Define a new function that is the area under the curve between $t=a$ and $t=x$.

$$F(x) = \int_a^x f(t)\,dt, \quad \text{where } x \in [a,b]$$

Fundamental Theorem of Calculus

$$\boxed{F'(x) = \frac{dF}{dx} = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$$

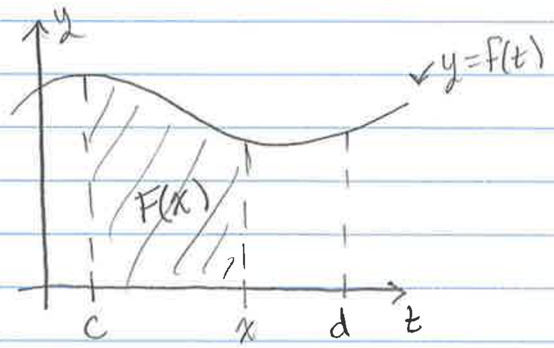
- every continuous f has an <u>antiderivative</u> F(x)

end video 7

- connection between derivatives and integrals (taking an integral is essentially taking an antiderivative).

Start video 8

Fundamental Theorem of Calculus 2

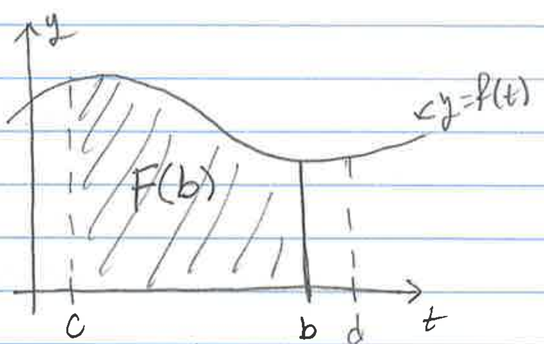


Fundamental theorem of calculus tells us that if

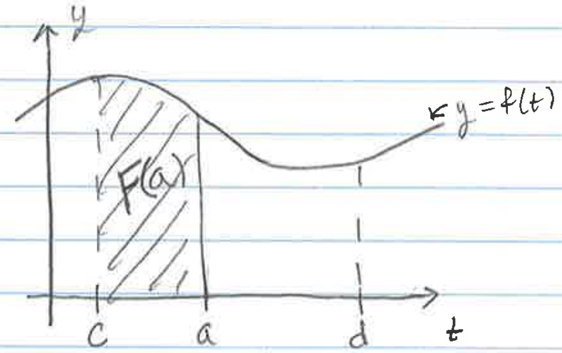
$$F(x)=\int_c^x f(t)\,dt$$

then $F'(x)=f(x)$. (F is the antiderivative of f)

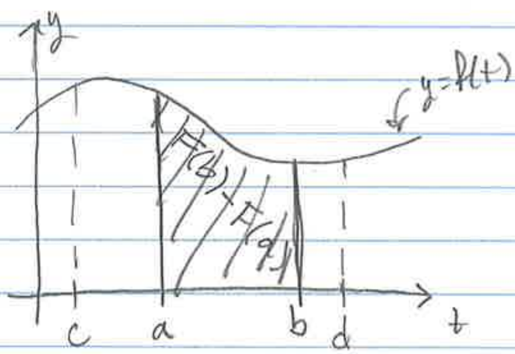
What is $F(b)-F(a)$?



$$F(b)=\int_c^b f(t)\,dt$$



$$F(a)=\int_c^a f(t)\,dt$$



$$F(b)-F(a)$$
$$=\int_c^b f(t)\,dt-\int_c^a f(t)\,dt$$
$$=\int_a^b f(t)\,dt$$

$$\boxed{\int_a^b f(t)\,dt=F(b)-F(a)}$$

where F is the antiderivative of f (i.e., $F'(x)=f(x)$)

end video 8