Multi-variable Calc.

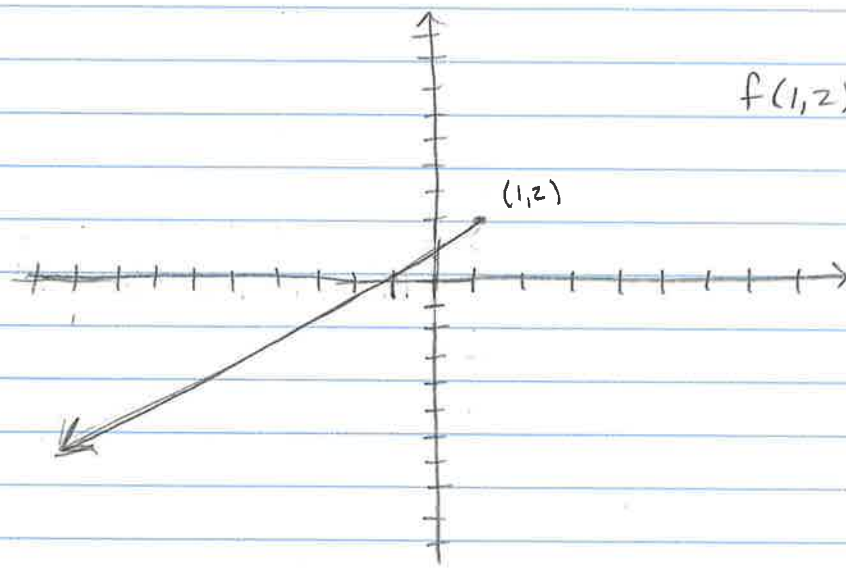
1

start video 9

Vector Fields

A way to visualize functions that have the same number of dimensions in its input as in its output.

Example: $f(x,y) = \begin{bmatrix} y^3 - 9y \\ x^3 - 9x \end{bmatrix}$



$f(1,2) = \begin{bmatrix} -10 \\ -8 \end{bmatrix}$

From each point in the input space, draw the output vector.

- To keep these plots looking clean, we typically scale down the lengths of the vectors and use colors or line widths to indicate the lengths.

end video 9

Video 10 has no notes, helpful for intuition though!

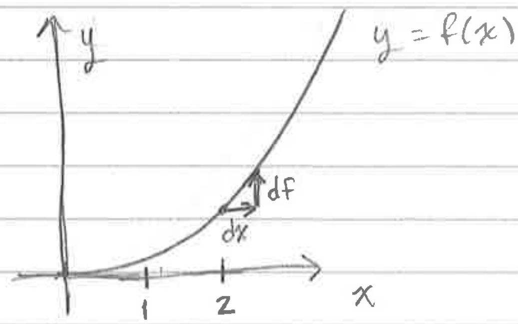
start video 11

Partial Derivatives

(secretly the same thing as ordinary derivatives)

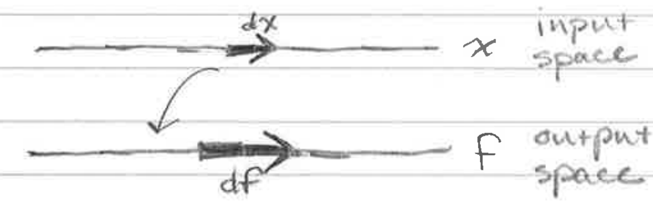
Consider $f(x) = x^2$,

$$f'(2) = \frac{df}{dx}(2) = 2x$$



dx : the size of a little nudge in the x direction

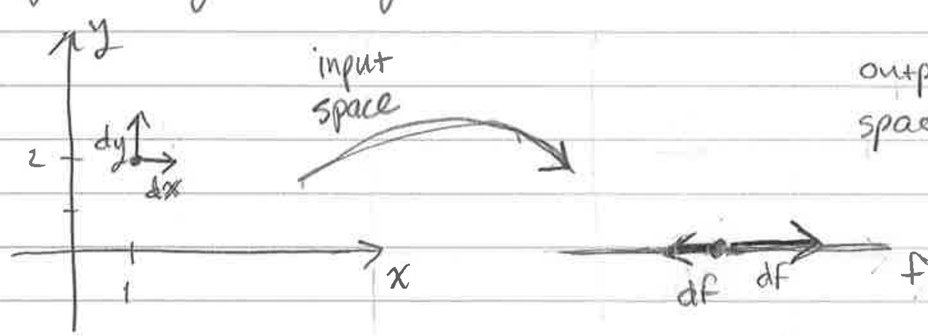
df: the resulting change in the output after you make that nudge



Now consider $f(x,y) = x^2 y + \sin(y)$

$$\frac{df}{dx}(1,2) = \frac{\partial f}{\partial x}(1,2)$$

$$\frac{df}{dy}(1,2) = \frac{\partial f}{\partial y}(1,2)$$



$\frac{\partial f}{\partial x}$: only cares about movement in the x direction, treats y as a constant.

$$\frac{\partial f}{\partial x}(1,2) = \frac{\partial}{\partial x}\left(x^2 \cdot 2 + \sin(2)\right)\Big|_{x=1} = 4x + 0\Big|_{x=1} = 4$$

evaluated at $x=1$

$$\frac{\partial f}{\partial y}(1,2) = \frac{\partial}{\partial y}\left(1^2 y + \sin(y)\right)\Big|_{y=2} = 1 + \cos(y)\Big|_{y=2} = 1 + \cos(y)$$

$f(x,y) = x^2y + \sin(y)$

$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(x^2y + \sin(y)\right) = 2xy + 0 = 2xy$

$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial y}\left(x^2y + \sin(y)\right) = x^2 + \cos(y)$

Because you're just looking at how the function changes in one direction, you treat the other variable as a constant and then take the ordinary derivative.

end video 11

Start video 12

Gradients

Consider $f(x,y) = x^2\sin(y)$

- $\frac{\partial f}{\partial x} = 2x\sin(y)$
- $\frac{\partial f}{\partial y} = x^2\cos(y)$

$$\Rightarrow \nabla f(x,y) = \begin{bmatrix} 2x\sin(y) \\ x^2\cos(y) \end{bmatrix}$$

The gradient of $f(x,y)$ is

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

Can think of

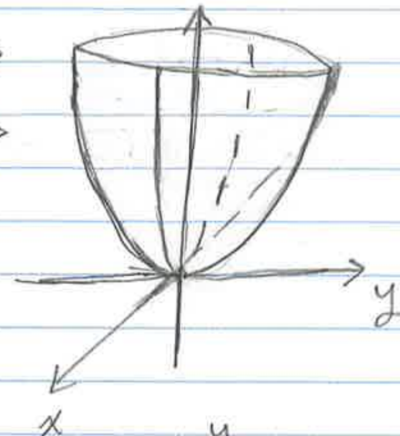
$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}, \quad \text{then} \quad \nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

end video 12

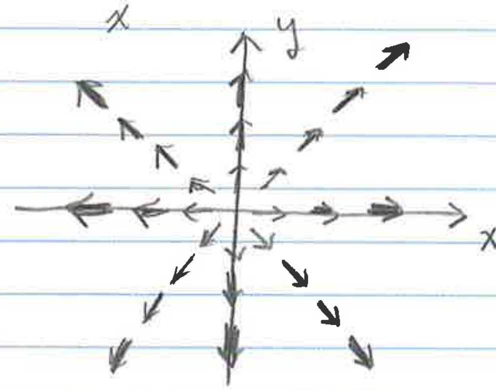
start video 13

Geometric Interpretation of Gradients

$$f(x,y) = x^2 + y^2$$



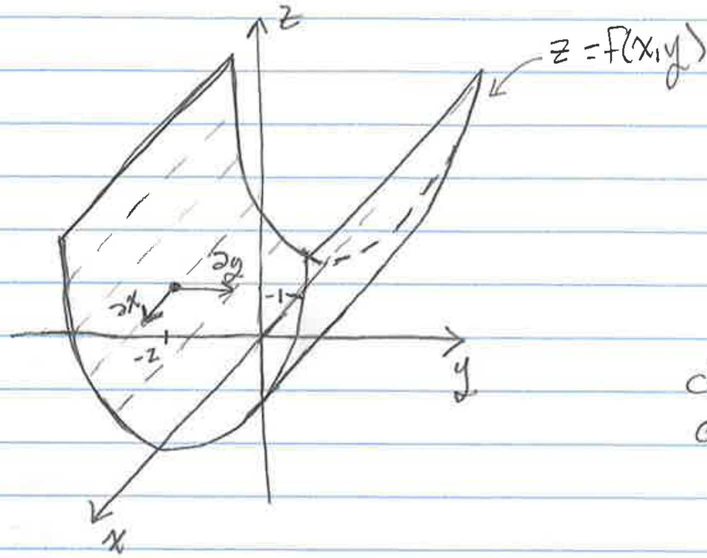
$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$



> **The gradient points in the direction of steepest ascent**

end video 13

Intuition on Gradients



changing x a small amount (∂x) caused no change in f

$$\frac{\partial f}{\partial x}(-1,-2) = 0$$

changing y a small amount (∂y) caused f to decrease

$$\frac{\partial f}{\partial y}(-1,-2) < 0$$

$$f(x,y) = y^2 \Rightarrow \nabla f(x,y) = \begin{bmatrix} 0 \\ 2y \end{bmatrix}$$

$$\Rightarrow \nabla f(-1,-2) = \begin{bmatrix} 0 \\ 2(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

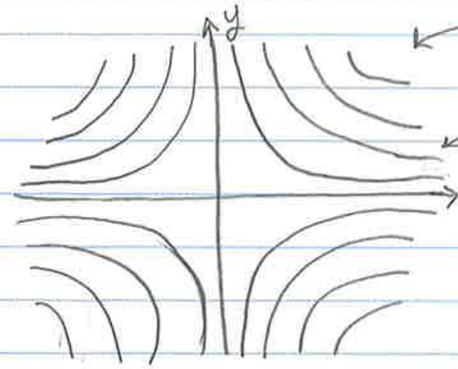
Gradient tells us not to move in the x direction at all, only move in the negative y direction from the point $(x,y) = (-1,-2)$.

- This is the direction of steepest ascent!

start video 14

Relationship between Gradients and Contour Lines

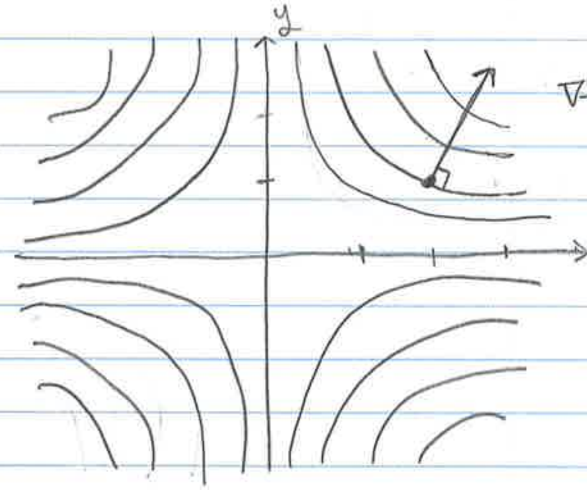
Consider $f(x,y) = xy$



each line represents when $f(x,y) = c$ for some constant c

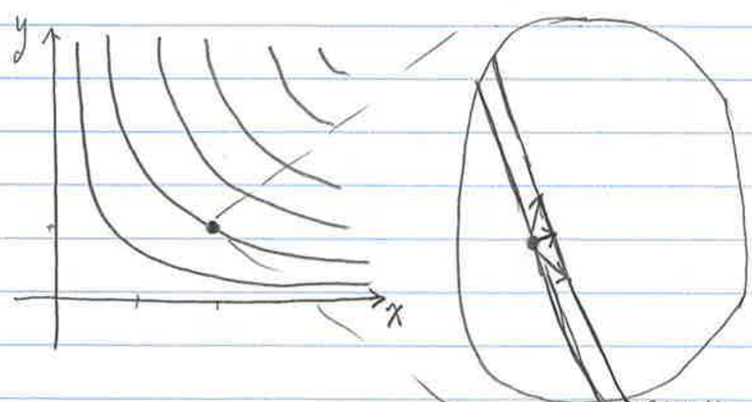
e.g. $f(x,y) = 2 \Rightarrow xy = 2$
$\Rightarrow y = 2/x$

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



$\boxed{\text{If the gradient vector is crossing a contour line it is perpendicular to that contour line!}}$

Recall: the gradient points in the direction of steepest <u>ascent</u>



Of all of the vectors that move from $f = 2$ to $f = 2.1$, which one does it the fastest (i.e., which one does it with the shortest distance)? The vector <u>perpendicular</u> to the contour line!

end video 14

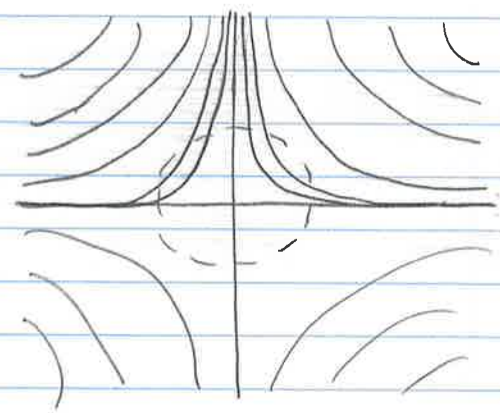
start video 15

Constrained Optimization Problems

Example: Maximize $f(x,y) = x^2 y$
on the set $\underbrace{x^2+y^2=1}_{\text{unit circle}}$

Looking on this circle projected onto the graph of $f(x,y)$ and looking for the highest points.

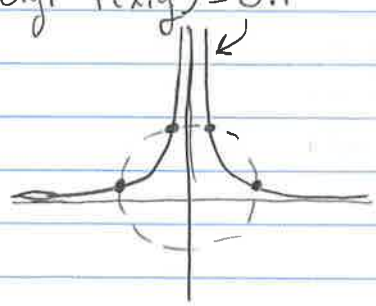
Let's limit our perspective to the input space, i.e, the contour map



solid lines = contour lines

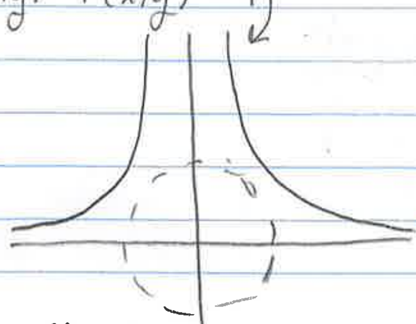
dotted line = unit circle (our constraint)

Some contour lines intersect with the circle (our constraint)
e.g. $f(x,y) = 0.1$



there are 4 pairs of #s (x,y) for which $f(x,y) = 0.1$ and the constraint is satisfied (i.e., $x^2+y^2=1$).

Other contour lines don't intersect with the circle
e.g. $f(x,y) = 1$

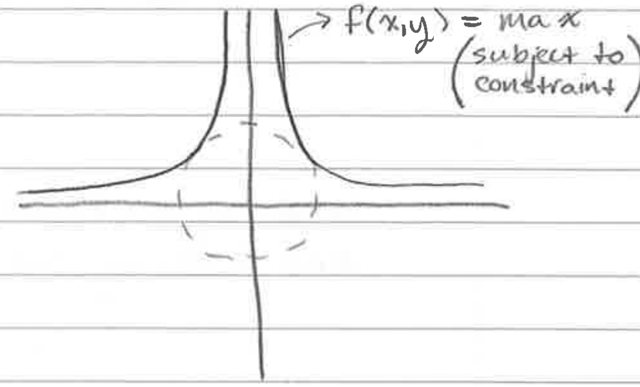


off $x^2+y^2=1$

there are no pairs of #s (x,y) for which $f(x,y)=1$ that will satisfy our constraint.

i.e., for all (x,y) such that $f(x,y)=1$, we have $x^2+y^2 \neq 1$.

We want to find the maximum value for $f(x,y)$ such that its contour line will still intersect w/ the circle (our constraint).



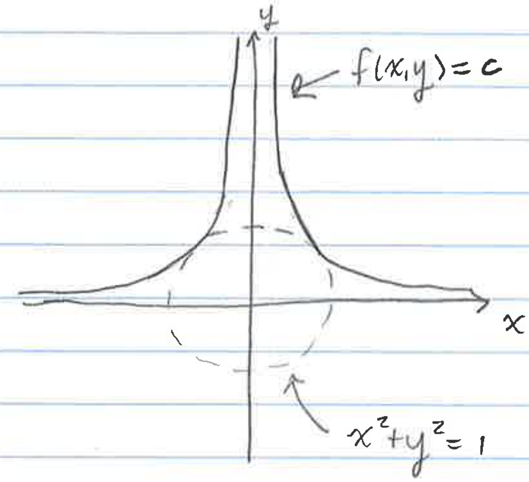
Key observation: the max value for $f(x,y)$ happens (that satisfies the constraint) when the contour line is tangent to the circle (to the constraint)

end video 15

Start video 16

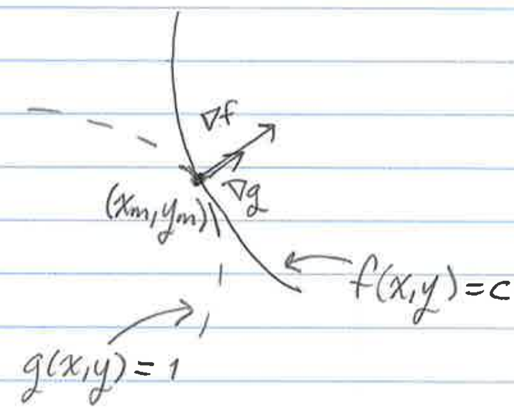
Lagrange Multipliers

Maximize $f(x,y) = x^2 y$
on the set $x^2 + y^2 = 1$.



This occurs when contour line $f(x,y) = c$ and the constraint $x^2+y^2=1$ are tangent.

Recall: every time the gradient passes through a contour line, it is perpendicular to it.



Let $g(x,y) = x^2+y^2$. Then our constraint $x^2+y^2=1$ is a contour line of $g(x,y)$ (i.e. $g(x,y)=1$).

So, ∇f perpendicular to $f(x,y)=c$
∇g perpendicular to $g(x,y)=1$
$\Rightarrow$ ∇f and ∇g are proportional to each other at the point of tangency $\underbrace{(x_m, y_m)}$ → this is also the point that maximizes f subject to our constraint!

$$\boxed{\nabla f(x_m, y_m) = \lambda \nabla g(x_m, y_m)}$$

"proportionality constant"
Lagrange multiplier

$$\nabla g = \nabla(x^2+y^2) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\nabla f = \nabla(x^2 y) = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}$$

Then $\nabla f(x,y) = \lambda \nabla g(x,y)$

$$\Rightarrow \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2xy &= \lambda 2x \\ x^2 &= \lambda 2y \end{aligned}\right\}$$

two equations, but three unknowns: x, y, and λ

$\Rightarrow$ need another equation. use constraint!

$x^2 + y^2 = 1$

$$\boxed{\begin{aligned} 2xy &= \lambda 2x \\ x^2 &= \lambda 2y \\ x^2+y^2 &= 1 \end{aligned}}$$

} what's necessary in order for our contour lines to be tangent to each other

} we have to be on the unit circle (i.e., we have to satisfy the constraint)

end video 16

start video 17

Need to solve the following system of equations

$$2xy = \lambda 2x$$
$$x^2 = \lambda 2y$$
$$x^2 + y^2 = 1$$

Assuming $x \neq 0$,

$$2xy = \lambda 2x$$
$$\Rightarrow 2y = \lambda 2$$
$$\Rightarrow y = \lambda$$

Use $y = \lambda$ in 2^{nd} equation,

$$x^2 = \lambda 2y$$
$$\Rightarrow x^2 = 2y^2$$

Use $x^2 = 2y^2$ in 3^{rd} equation

$$x^2 + y^2 = 1$$
$$\Rightarrow 2y^2 + y^2 = 1$$
$$\Rightarrow 3y^2 = 1$$
$$\Rightarrow y^2 = \frac{1}{3}$$
$$\Rightarrow \boxed{y = \pm\sqrt{\frac{1}{3}}}$$

From $x^2 = 2y^2$

$$\Rightarrow x^2 = 2\left(\frac{1}{3}\right)$$
$$\Rightarrow \boxed{x = \pm\sqrt{\frac{2}{3}}}$$

We assumed $x \neq 0$, so now we need to consider what happens if $x = 0$.

If $x = 0$ then $x^2 + y^2 = 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

Then the 2^{nd} equation gives $x^2 = \lambda 2y = \pm 2\lambda$

but $x = 0$, so $x^2 = 0 \Rightarrow 0 = \pm 2\lambda$

$$\Rightarrow \lambda = 0$$

but λ is a proportionality constant so it can't be 0!

This means then that x can't be 0 either $(x \neq 0)$.

So, we must have $\boxed{x = \pm\sqrt{\frac{2}{3}}, \quad y = \pm\sqrt{\frac{1}{3}}}$

So, there are four points that could potentially maximize $f(x,y) = x^2y$ under the constraint $x^2+y^2=1$

$$\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right), \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right), \left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right), \left(-\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right)$$

Plug potential points into $f(x,y) = x^2y$ to find which makes $f(x,y)$ the largest

Recall x^2 is always positive, so plugging a negative value for y make's the entire function negative, whereas plugging in a positive value for y will make the function positive

$\Rightarrow \left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right)$ and $\left(-\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right)$

will <u>not</u> make $f(x,y)$ as large as it can be

Instead check $\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right), \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right)$

$$f\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) = \left(\sqrt{\frac{2}{3}}\right)^2\sqrt{\frac{1}{3}} = \frac{2}{3}\sqrt{\frac{1}{3}}$$

$$f\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) = \left(-\sqrt{\frac{2}{3}}\right)^2\sqrt{\frac{1}{3}} = \frac{2}{3}\sqrt{\frac{1}{3}}$$

$\Rightarrow$ Both $\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right)$ and $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right)$ will maximize $f(x,y)$ subject to the constraint $x^2+y^2=1$.

end video 17