EIGENVALUES / LINEAR DIFFEQ | (pre-class notes) page 1



## *COMPLEX NUMBERS

$$x^2 + 1 = 0 \qquad x = ?$$

$$x^2 = -1$$

$$\Rightarrow \quad x = \pm\sqrt{-1} \qquad \text{Define } i = \sqrt{-1}$$

what about $\quad x^4 - 1 = 0 \qquad x = ?$

## *FUNDAMENTAL THM OF ALGEBRA

Every real or complex polynomial of degree "n" has "n" roots (can be complex AND repeated)

$$\sqrt{x^4} = \sqrt{1} \Rightarrow \sqrt{x^2} = \sqrt{\pm 1} \Rightarrow x = \pm i$$

$$x = \{+1, -1, +i, -i\} \leftarrow 4 \text{ roots}$$

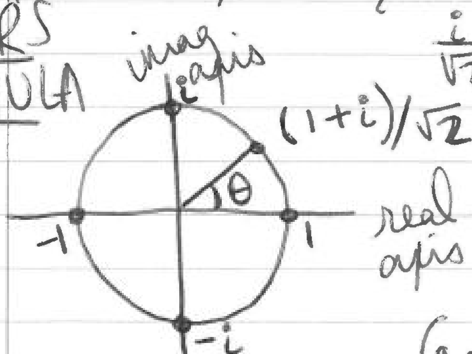
$x^8 = 1 \leftarrow$ we will need the square root of $i$ (!)

let $\omega = (1+i)/\sqrt{2}$

$$\Rightarrow \omega^2 = \tfrac{1}{2}(1+i)^2 = \tfrac{1}{2}(1 + 2i + \overset{-1}{i^2}) = i$$

$$x = \{+1, -1, i, -i, \tfrac{1}{\sqrt{2}}(1+i), -\tfrac{1}{\sqrt{2}}(1+i), \tfrac{i}{\sqrt{2}}(1+i), \tfrac{-i}{\sqrt{2}}(1+i)\}$$

EULER'S FORMULA




$$(1+i)/\sqrt{2} = \cos\theta + i\sin\theta = e^{i\theta}$$

$$(e^{i\theta})(e^{-i\theta}) \overset{?}{=} 1$$

$$(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$$

$$\underbrace{\cos^2\theta + \sin^2\theta}_{=1} + i\sin\theta\cos\theta \cancel{- i\sin\theta\cos\theta}$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots = \begin{array}{l}\cos\theta = 1 - \frac{\theta^2}{2} + \dots \\ + i\sin\theta = i(\theta - \frac{\theta^3}{6} + \dots)\end{array}$$

## * EIGENVALUES & EIGENVECTORS

MATRIX $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

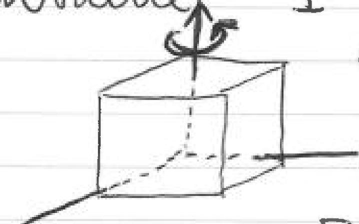
find vectors that stay on their own span!

e.g. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

EX: Consider a 3D rotation. The eigenvector of that rotation is the AXIS OF ROTATION w/ eigenvalue = 1



e.g. rotation in xy around z:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is eigenvector w/ eigenvalue 1

DEF: $A\vec{v} = \lambda\vec{v}$ $\quad$ $\lambda$: eigenvalue, $\vec{v}$: eigenvector

rewrite as

$A\vec{v} = \lambda I \vec{v}$ $\quad$ identity

$\Rightarrow A\vec{v} - \lambda I\vec{v} = \vec{0} \Rightarrow (A - \lambda I)\vec{v} = \vec{0}$

ONE SOLUTION $\vec{v} = \vec{0}$ (trivial solution)

*ONLY OTHER WAY TO BE ZERO IS IF $\det(A - \lambda I) = 0$ *

Ex: $\det\left(\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{vmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(2-\lambda) = 0$

$\Rightarrow \lambda = \{3, 2\}$

## * EIGENVALUES & EIGENVECTORS (CONT.)

Ex: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ← rotation matrix

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

All vectors in the REAL plane are rotated ⇒ no REAL vectors that stay in their own span

Ex: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda = 1$$

only ONE eigenvalue / eigenvector

eigenvector: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 1\begin{bmatrix} a \\ b \end{bmatrix}$

$$a + b = a \Rightarrow b = 0$$

$$\cancel{b = b} \Rightarrow a \in \mathbb{R}$$

eigenvector $\vec{v} = \begin{bmatrix} a \\ 0 \end{bmatrix}$

Ex: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)(2-\lambda)$$

$$\Rightarrow \lambda = 2$$

eigenvectors: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 2\begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{matrix} 2a = 2a \\ 2b = 2b \end{matrix} \Rightarrow a, b \in \mathbb{R}$$

all vectors are eigenvectors

END @ 13:00

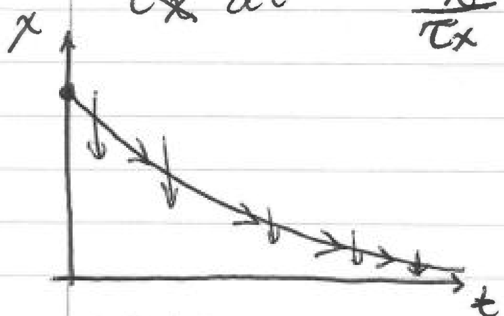
## * DIFFERENTIAL EQUATIONS

↑ "SYSTEMS OF EQUATIONS" GOVERNING DYNAMICS

use linear algebra (eigenvalues / eigenvectors)

$x$ = firing rate of a neuron, $\tau_x$ = timescale of neuron

$$\tau_x \frac{dx}{dt} = -\frac{x}{\tau_x} \quad \Big\} \text{ firing rate decays to zero}$$




| $x$ | $dx/dt$ |
|---|---|
| 2 | $-2/\tau_x$ |
| 1 | $-1/\tau_x$ |
| 0.1 | $-0.1/\tau_x$ |

$$\frac{dx}{dt} = -\frac{x}{\tau_x}$$

$$\Rightarrow \int \frac{dx}{x} = \int -\frac{dt}{\tau_x}$$

$$\Rightarrow \ln x = -t/\tau_x + c$$

$$\Rightarrow x = e^{-t/\tau_x + c}$$

$$x = c e^{-t/\tau_x}$$

$\underbrace{\text{(what happens if } x \text{ starts @ 0)}}_{0 \text{ is a fixed point}}$

$\Rightarrow$ neuron's firing rate decays w/ timescale $\tau_x$

Now let's give the neuron input from another neuron:

$$\left.\begin{aligned} \frac{dx}{dt} &= -x + 2y \\ \frac{dy}{dt} &= -y + 2x \end{aligned}\right\} \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Suppose the solution takes the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} \Rightarrow \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}(a_1 e^{\lambda t}) \\ \frac{d}{dt}(a_2 e^{\lambda t}) \end{bmatrix} = \begin{bmatrix} \lambda a_1 e^{\lambda t} \\ \lambda a_2 e^{\lambda t} \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cancel{e^{\lambda t}} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cancel{e^{\lambda t}}$$

$\lambda$ is eigenvalue & $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is eigenvector

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ Find eigenvalues & eigenvectors

Eigenvalues

$$\det\left(\begin{bmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}\right) = (+1+\lambda)(+1+\lambda) - 4 = 0$$

$$= \lambda^2 + 2\lambda \overset{-3}{\cancel{+1 - 4}} = 0$$

$$= (\lambda + 3)(\lambda -1) = 0$$

$$\Rightarrow \lambda = \{3, +1\}$$

Eigenvectors

① $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -3 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$-a_1 + 2a_2 = -3a_1 \Rightarrow a_2 = -a_1 \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Check $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 - 2 \\ 2 + 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \overset{\checkmark}{=} -3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

② $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 1 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$-a_1 + 2a_2 = a_1 \Rightarrow a_2 = a_1 \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Check $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 + 2 \\ 2 - 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

SOLUTION: $\underbrace{\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}}$

can add equations and still have solution
(linear) combinations are also solutions *

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

Check of solution

RHS $\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} -3c_1 e^{-3t} + c_2 e^{t} \\ +3c_1 e^{-3t} + c_2 e^{t} \end{bmatrix}$

LHS $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{-3t} + c_2 e^{t} \\ -c_1 e^{-3t} + c_2 e^{t} \end{bmatrix} = \begin{bmatrix} -c_1 e^{-3t} - c_2 e^{t} - 2c_1 e^{-3t} + 2c_2 e^{t} \\ 2c_1 e^{-3t} + 2c_2 e^{t} + c_1 e^{-3t} - c_2 e^{t} \end{bmatrix}$

$$= \begin{bmatrix} -3c_1 e^{-3t} + c_2 e^{t} \\ 3c_1 e^{-3t} + c_2 e^{t} \end{bmatrix} \checkmark$$

$$\boxed{\begin{aligned} dx/dt &= -x + 2y \\ dy/dt &= 2x - y \end{aligned}}$$

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \lambda_1 = -3,\ \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix};\ \lambda_2 = 1,\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If you start on an eigenvector, STAY on e'vector

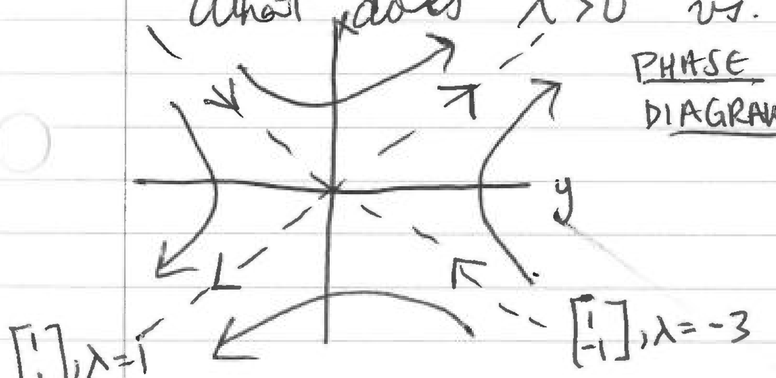
"PROOF": Euler's method

$$\begin{bmatrix} x(t+\Delta t) \\ y(t+\Delta t) \end{bmatrix} \approx \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \Delta t \begin{bmatrix} dx(t)/dt \\ dy(t)/dt \end{bmatrix}$$

$$\text{if } \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \vec{v} \text{ e'vector} \Rightarrow \begin{bmatrix} dx(t)/dt \\ dy(t)/dt \end{bmatrix} = \lambda \vec{v}$$

$$\Rightarrow \begin{bmatrix} x(t+\Delta t) \\ y(t+\Delta t) \end{bmatrix} \approx \vec{v} + \Delta t \lambda \vec{v} = \underbrace{(1 + \Delta t \lambda)\vec{v}}_{\text{still on e'vector}}$$

What does $\lambda > 0$ vs. $\lambda < 0$ mean?




PHASE DIAGRAM

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

From MOST initial conditions

$x, y \to \pm\infty$

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How can we make this system stable?

SIMPLE SOLUTION:

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \Bigg( \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}}_{A} \underbrace{- 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{-2I} \Bigg) \begin{bmatrix} x \\ y \end{bmatrix}$$

make neurons decay faster, why does this work?

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \det(A - 2I - \lambda_{new} I) \stackrel{?}{=} \det(A - (2 + \lambda_{new}) I) = 0$$

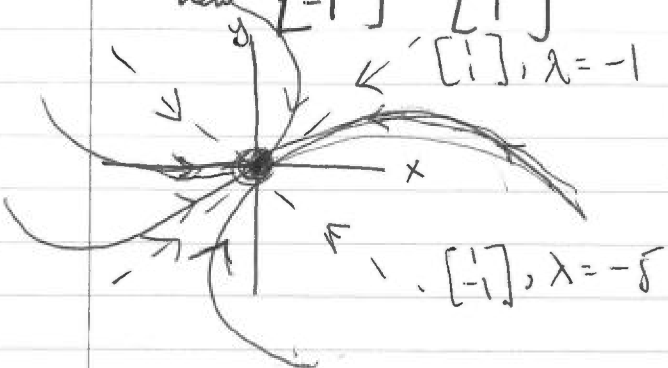
$$\Rightarrow 2 + \lambda_{new} = \lambda \Rightarrow \boxed{\lambda_{new} = \lambda - 2}$$

THM: Eigenvalues of $A + bI = \lambda + b$
where $\lambda$ are eigenvalues of $A$
and eigenvectors same $(A + bI)\vec{v}_{new} = (\lambda + b)\vec{v}_{new}$

$$A\vec{v} + bI\vec{v} = \lambda\vec{v} + b\vec{v} \Rightarrow \vec{v} \text{ satisfies equation}$$

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \lambda_{new} = \{-5, -1\}$$

$\vec{v}_{new} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ same as $\vec{v}$ (check to see if true on your own)




$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} \xrightarrow[t \to \infty]{} 0$$

How else can we have neurons $\not\to \infty$?

– Add inhibitory neurons!

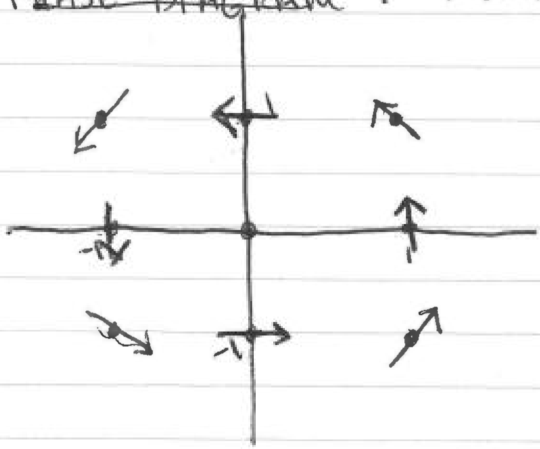
$x$ = excitatory
$y$ = inhibitory

$$\begin{aligned} dx/dt &= -y \\ dy/dt &= +x \end{aligned} \Rightarrow \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find e'values: $\det\left( \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \right) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm\sqrt{-1} = \pm i$

so $x$ + $y$ are functions of $e^{it} = \cos t + i \sin t$

~~PHASE DIAGRAM~~: Find $dx/dt$, $dy/dt$



| $x$ | $y$ | $dx/dt = -y$ | $dy/dt = x$ |
|---|---|---|---|
| 0 | 1 | −1 | 0 |
| 1 | 0 | 0 | 1 |
| −1 | 0 | 0 | −1 |
| 0 | −1 | 1 | 0 |
| 1 | 1 | −1 | 1 |
| 1 | −1 | 1 | 1 |
| −1 | 1 | −1 | −1 |
| −1 | −1 | 1 | −1 |

Can anyone guess what the solutions look like?

* what happens when $\lambda = -1 \pm i$