COMBINATORICS

Ex. 1

HOW MANY WAYS ARE THERE TO PLACE 3 BALLS IN 2 BOXES, EACH CONTAINING 1 BALL MAXIMUM?

A B C → |_|_|

LET'S COUNT:

|A|B| |B|A| |B|C| |C|B| |A|C| |C|A| |A|A| |B|B| |C|C|

WE WILL DISTINGUISH BASED ON THE ORDER IN THIS CASE

WE HAVE SEVERAL COPIES OF EACH BALL

$$S = \{A, B, C\} \quad \text{3 ELEMENTS}$$

A, B, C = 3 → |_|_| → |A|_| ← A, B, C = 3

ORDERED COMBINATIONS WITH REPLACEMENT $= 3 \cdot 3 = 3^2$

n^r

r → number of boxes

n → number of elements

Ex. 2

WHAT IF WE ONLY HAVE 1 COPY OF EACH ELEMENT?

A, B, C = 3 B, C = 2

$\longrightarrow \quad 3 \cdot 2 = 6$

GENERAL CASE

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) = n \cdot (n-1) \cdots (n-r+1) \cdot \frac{(n-r)(n-r-1) \cdots 1}{(n-r)(n-r-1) \cdots 1}$$

$$= \frac{n!}{(n-r)!} \qquad \boxed{r \leq n}$$

ORDERED COMBINATIONS WITHOUT REPLACEMENT

IF $r = n \longrightarrow$ PERMUTATIONS

Ex. 3

WHAT IF WE DON'T CARE ABOUT THE ORDER?

-> |A|B| and |B|A| COUNT AS ONE

UNORDERED COMBINATIONS WITHOUT REPLACEMENT

CASE $n = 3$, $r = 3$

|A|B|C| |A|C|B| |B|A|C| |B|C|A| |C|A|B| |C|B|A|

$3 \cdot 2 \cdot 1 \longrightarrow$ IN GENERAL $= r(r-1) \cdots 1 = r!$

$$\frac{n!}{(n-r)!} \longrightarrow \frac{n!}{(n-r)!\, r!} \equiv \binom{n}{r} \equiv \binom{n}{n-r}$$

BINOMIAL COEFFICIENT

WHY IS IT CALLED BINOMIAL?

BINOMIAL FORMULA

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Ex.

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b) =$$

- 1 WAY TO COMBINE ALL as $\rightarrow a^4$
- 4 WAYS TO COMBINE 3 as AND 1 b $\rightarrow 4a^3b$
- ETC...

↳ EQUIVALENT TO PLACING 1 "b" BALL IN 4 BOXES

$$= a^4 + b^4 + 4a^3b + 6a^2b^2 + 4ab^3$$

$$\downarrow$$

$$\frac{4!}{(4-1)!\,1!} = \frac{4!}{3!} = 4$$